Studying the inertia of an LCM matrix

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Abstract

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers with $x_i \leq x_j \Rightarrow i \leq j$. The GCD matrix (S) of the set S is the $n \times n$ matrix with $\gcd(x_i, xj)$ as its ij entry. Similarly, the LCM matrix $[S]$ of the set S has $\mathrm{lcm}(x_i, x_j)$ as its ij entry. Both of these matrices were originally defined by H. J. S. Smith in his seminal paper [4] from the year 1876.

During the last 30 years both GCD and LCM matrices (as well as their various generalizations) have been investigated extensively in the literature. However, GCD matrices are in many ways easier to study than LCM matrices. For example, the GCD matrix (S) is positive definite for any set S whereas the LCM matrix $[S]$ is almost always indefinite and may be even singular. Very little is known about the inertia of the matrix $[S]$ in general. One can of course make some additional assumptions about the set S, but still the matrix $[S]$ remains quite hard to study. In 1992 Bourque and Ligh [1] conjectured that if the set S is GCD closed (that is, $\gcd(x_i, x_j) \in S$ for all $x_i, x_j \in S$), then the matrix $[S]$ is nonsingular. A few years later it was shown that this conjecture holds only for GCD closed sets with at most 7 elements, but not in general for larger sets (see [2] and [3]).

It turns out that if the set S is GCD closed, then the poset-theoretic semilattice structure of $(S, |)$ often alone determines the inertia of the LCM matrix $[S]$ completely. This is a bit surprising, since one could expect the exact values of the elements $x_i \in S$ to play a bigger role in this. In this presentation we are going to define a new lattice theoretic concept and use it to give an explanation to this mystery. We also show some examples how to determine the inertia of the matrix $[S]$ by looking only at the semilattice structure of $(S, |)$. At the same time we are able to give an elegant proof to the well-known result that the Bourque-Ligh conjecture holds for all except for one GCD closed set with at most 8 elements.

Keywords

LCM matrix, GCD matrix, inertia, eigenvalue, Bourque-Ligh conjecture, Möbius inversion.

References

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