A Branch-and-Bound scheme for the range of rank-deficient quadratic forms with interval-valued variables

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Abstract

Given a quadratic form $f(x) = x^T Qx$ and bounds $\underline{x} \leq x \leq \overline{x}$ for its variables, we address the problem of computing the range $\underline{f} = \min_{\underline{x} \leq x \leq \overline{x}} f(x)$ and $\overline{f} = \max_{\underline{x} \leq x \leq \overline{x}} f(x)$. First we address the case when Q is positive semidefinite. Then the lower bound $\underline{f}$ can be computed efficiently via CQP, while computation of the upper bound $\overline{f}$ is NP-hard. We focus on the case when Q is rank-deficient. We reformulate the computation of $\overline{f}$ as a problem of enumeration of vertices of a zonotope in d-dimensional space [4], where $d = \text{rank}(Q)$. Instead of constructing the enumeration of vertices in full (as in [1, 5]), we design a B&B scheme. The branching step consists in a split of a zonotope into a pair of "smaller" zonotopes by removal of a generator. In the bound-part, we use Goffin's method [2] to approximate a zonotope by a pair of Löwner-John ellipsoids. Then, the lower and upper bound for f over an ellipsoid is computed by Ye's algorithm [6] for optimization of (arbitrary) quadratic forms over ellipsoids. We also discuss the impact of various strategies for the choice of (i) the active zonotope, (ii) the branching generator and (iii) the method for computation of lower bounds.

The general case, when Q need not be positive semidefinite (but is still rank-deficient), can be reduced to the problem of enumeration of all k-dimensional faces of a $2d$-dimensional zonotope, where $k = 0, 1, \ldots, 2d$. This can be done by zonotope enumeration algoritms [3]. We design a B&B strategy for this case, too. Here, the branching step consists in replacement of a zonotope by a pair of zonotopes with a shorter generator. The bounding step is similar to the psd case. This B&B scheme generates a potentially infinite branching tree with a branch converging to the maximizer/minimizer. Cutting the tree at a certain level allows us to compute an ε-approximate solution.
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Keywords

Quadratic form, Interval data, Zonotope, Brach and Bound.

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